Experimental Studies of Three-Dimensional Filtration on a Circular Leaf

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A previous study (3) revealed a discrepancy between the actual shape of a spheroidal filter cake formed on a laterally unconfined circular test leaf, and that predicted from theory. This discrepancy was tentatively attributed to either cake compressibility or filter cloth resistance that was neglected in the mathematical model and not controlled in the original experiments. More extensive and systematic experiments reported in the present paper, employing incompressible cakes and filter media of negligible resistance, display similar behavior. (Cake incompressibility and negligible septum resistance were quantitatively established by conventional one-dimensional experiments.) It is speculated that the observed departure from theory may be due to the inapplicability of Darcy's law, especially in the early stages of cake formation.

Until recently (3) filtration theory dealt primarily with filter-cake growth in one-dimension, that is, where cake buildup occurs in such a manner that the cake surface always lies parallel to that of the filter cloth. This type of filtration, widely used in industrial applications, is theoretically described by a one-dimensional filtration rate equation as expressed by the familiar Ruth equation (*1*). A more general filtration theory, however, should describe the growth of a cake uninhibited by lateral constraints. The cake so deposited no longer grows entirely normal to the cloth; it grows laterally as well. Such a process is described as a three-dimensional filtration.*

Experimentally and theoretically, the simplest example of this phenomenon occurs when a slurry is filtered on a circular filter cloth (Figure 1). The driving force for the process is provided by creating a fixed vacuum on the underside of the cloth, atmospheric pressure being maintained at the cake surface. The pressure drop across the cake thus remains constant throughout the filtration. A thin, flat O-ring of large outside diameter is placed around the cloth to insure that the base of the resulting cake lies in the same plane as the cloth. Due to geometric symmetry, any cake so grown will be spheroidal in shape, possessing a circular cross section. One obvious effect of this three-dimensional growth is to provide an ever-increasing mean area for filtrate flow, due to continuous enlargement of the outer filter-cake surface. This results in a smaller resistance to filtrate flow and, hence, a greater rate of filtration than would be observed in the corresponding one-dimensional filtration experiment performed with a laterally restrained circular test leaf of equal filter cloth area.

The work summarized here, and described in greater detail elsewhere (*4*), represents an attempt to delineate the limits of validity of the major aspects of the original theory (3).

* It should be clearly understood that filtration on, for instance, a *spherical* test leaf of the type discussed in the appendix to reference 3 is one-dimensional. Since the streamlines for filtrate flow through the spherical cake are necessarily straight lines, the cake will be spherical in shape at any stage of the filtration process, irrespective of such factors as cake *compressibility, septum resistance, or the applicability* of Darcy's law. An essential prerequisite for three-dimensional cake growth (in the sense employed in this paper) is that the streamlines be curved. Only then is the cake profile influenced by these factors.

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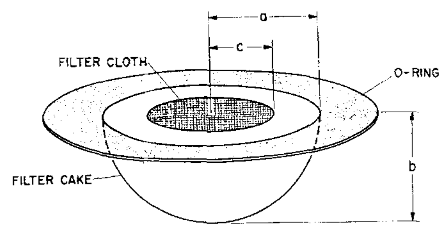

Fig. 1. Growth of a spheroidal cake on a circular leaf.

PRIOR WORK

Brenner (3) made a theoretical analysis of this three-dimensional phenomenon; developed the basic filtration rate equation, $V = V(t)$, for the growth of a three-dimensional cake on a circular leaf; and demonstrated that certain phenomena arise which have no counterpart in conventional one-dimensional filtrations. The major results of that analysis will be reviewed to provide background for the experimental methods used for its attempted verification.

Consider the filtration of a homogeneous suspension on one side of a flat, circular filter cloth (Figure 1). A constant vacuum (absolute pressure p_t) is applied to the underside of the cloth surface; the outer cake surface, in contact with the slurry, is essentially at atmospheric pressure p_o (neglecting minor differences in hydrostatic pressure due to the depth of the slurry). This results in a constant pressure drop $\Delta p = p_o - p_t$ across the cake. At any stage in the cake growth, its equatorial radius is denoted by a and its thickness (polar radius) by b.

The assumptions used to simplify the analysis were as follows: (1) the filter cake formed is incompressible; (2) the filter cloth offers negligible resistance to the flow of filtrate; and (3) Darcy's law is valid:

$$\vec{v} = -\frac{k\,g_c}{\mu}\nabla p \tag{1}$$

By combining this relation with the continuity equation for incompressible flow, and employing the boundary conditions imposed on the pressure field, one can obtain the following differential equation, which describes the instantaneous rate of filtrate flow through the cloth:

$$q = \frac{dV}{dt} = \frac{2\,\pi\,ck\,\Delta p g_c}{\mu\,\tan^{-1}\xi_o} \tag{2}$$

where

$$\xi_o = b/c \tag{3}$$

By means of a material balance on the solids in the slurry and cake, one obtains another relation between V and ξ_o [see Equation (4)]. Substitution of the latter into Equation (2), followed by integration subject to the initial condition of no cake at the start of the experiment, ultimately leads to the basic filtration relationship, $V = V(t)$, in the parametric form

$$\left\{ \frac{3\,\rho w}{2\,\pi\,c^3\,(1-mw)\,(1-\epsilon)\rho_s} \right\} V = \xi_o\,(1+\xi_o^2) \tag{4}$$

in which the parameter ξ_o is to be obtained from the relation

$$\xi_o\,(1+\xi_o^2)\tan^{-1}\xi_o - \frac{1}{2}\xi_o^2 = \left\{ \frac{3\,k\,\Delta p\,g_c\,\rho w}{c^2\,\mu\,(1-mw)\,(1-\epsilon)\rho_s} \right\} t \tag{5}$$

The parameter $\xi_o = \xi_o(t)$ represents the dimensionless "thickness" of the filter cake that increases with time. With the exception of V, all terms appearing in the left-hand side of Equation (4) remain constant during any one run. The left side of Equation (4) may therefore be thought of as a dimensionless cumulative filtrate volume $\mathcal{V}$. Similarly, with the exception of t, all quantities on the right-hand side of Equation (5) remain constant during any single run, and the entire grouping of terms on the right side may be regarded as a dimensionless time $\mathcal{T}$. By eliminating the parameter ξ_o between Equations (4) and (5) one obtains the basic filtration equation $\mathcal{V} = \mathcal{V}(\mathcal{T})$ or, what is equivalent, $V = V(t)$ for any given run. These equations must necessarily remain in implicit parametric form, for, owing to the transcendental nature of the arctangent, no explicit solution is possible.

By utilizing the expansions $\tan^{-1}\xi_o = \xi_o + O(\xi_o^3)$ for $\xi_o << 1$, and $\tan^{-1}\xi_o = \pi/2 + O(\xi_o^{-2})$ for $\xi_o >> 1$ one can, however, obtain limiting formulas for long and short filtration times in explicit form. Thus, from Equations (4) and (5), one finds for short times

$\xi_o \to 0$ [i.e. $t << c^2\,\mu\,(1-mw)\,(1-\epsilon)\,\rho_s/3k\,\Delta p\,g_c\,\rho w$], that

$$V \to \pi\,c^2 \sqrt{\frac{8k\,\Delta p\,g_c\,\rho_s\,(1-\epsilon)\,(1-mw)\,t}{3\,\mu\,\rho w}} \tag{6}$$

whereas for long times

$\xi_o \to \infty$ [i.e. $t >> c^2\,\mu\,(1-mw)\,(1-\epsilon)\,\rho_s/3k\,\Delta p\,g_c\,\rho w$],

$$V \to \frac{4ck\,\Delta p\,g_c\,t}{\mu} \tag{7}$$

The corresponding formula for one-dimensional filtration on a laterally constrained test leaf of radius c is, under comparable conditions (3)

$$V = \pi c^2 \sqrt{\frac{2k\,\Delta p\,g_c\,\rho_s\,(1-\epsilon)\,(1-mw)\,t}{\mu\,\rho w}} \tag{8}$$

This relation holds for all times t. By comparing Equations (6) and (8) one sees that, apart from a constant factor* of $\sqrt{4/3}$, the behavior at small times is essentially one-dimensional. However, as is clear from Equation (7), for long times the behavior is very different. In particular, the flow rate does not fall to zero as the cake thickness increases indefinitely. Rather, a limiting flow rate is approached.

According to the theory (3), the outer surface of the cake at any stage of growth is described by the semiellipsoid of revolution

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1 \quad (z > 0) \tag{9}$$

having the cloth as the focal circle. The equatorial and polar radii a and b are thus related by the expression

$$a^2 - b^2 = c^2 \tag{10}$$

where c remains constant throughout the experiments.

The few preliminary experiments reported in reference 3 revealed significant departures from the theoretical predictions, especially in the early stages of cake formation. In particular, the spheroidal cakes were observed to be much flatter than predicted by Equation (10). However, no serious efforts were made at the time to achieve the conditions of cake incompressibility or negligible cloth resistance required for a truly rigorous test of the theory. As a result the discrepancy was tentatively attributed to either the compressibility of the cake or, less likely, to the resistance offered by the filter cloth. The following experiments were designed primarily to test these hypotheses. Both three-dimensional and conventional one-dimensional experiments were conducted, the latter being required to establish quantitatively the degree of compressibility of the filter cakes and the magnitude of the cloth resistance.

THREE-DIMENSIONAL TESTS

Figure 2 depicts the constant pressure vacuum filtration system employed in these tests. The filter leaf itself is shown in Figure 3. It was constructed from a "standard" 1/10 sq.ft. circular test leaf (Dorr-Oliver) by fastening a Lucite O-ring of 10-in. O.D. around the 4-in. diameter filter cloth. Appropriate gaskets and fasteners insured water tightness everywhere except, of course, at the cloth itself. In those tests which required a smaller effective filter area, the same leaf was modified by partially covering the filter cloth with a Lucite O-ring of 1.33-in. I.D., thus sealing off the outer rim of the cloth. The slurry was contained in a 12-gal. polyethylene baffled tank equipped with a mixer, and the clear filtrate was collected in a 5-gal. glass receiver calibrated in 0.5-liter divisions.

Since earlier tests (3) suggested a compressibility effect, these tests were done with an incompressible material in order to settle the matter. To meet the requirements of cake incompressibility and low sedimentation rate, Lucite 4F molding powder was dispersed (with the help of 0.04 wt. % Triton X-100) in a 40 wt. % aqueous sugar solution, to which a trace of formaldehyde was added as a preservative. The particles comprising this molding powder are uniformly spherical in shape; 75% of their diameters lie between 40 and 150μ. Their true spe-

* The reason for the unexpected appearance of this factor is discussed in reference 3.

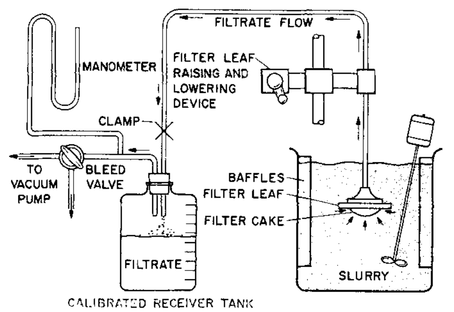

Fig. 2. Flow diagram for three-dimensional filtration.

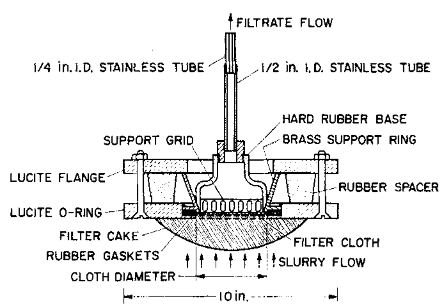

Fig. 3. Test leaf for three-dimensional filtration.

cific gravity is 1.18, identical to that of the sugar solution. Hence, only intermittent stirring was required to maintain a homogeneous dispersion. Exact specific gravity of the sugar solution was determined by refractometry, using available (2) refractive index-density correlations. This was double checked via a Westphal balance, thus permitting the densities to be accurately matched. Viscosities were obtained from reference 6 and experimentally verified with a Cannon-Fenske viscometer. The Lucite concentration varied from 7 to 10% by weight during the course of the three-dimensional investigation, though it was constant during any single run.

In practice, the filter leaf was used in an inverted position. This was purely a matter of convenience, resulting from the physical configuration of the equipment. Since the Lucite particles exhibited negligible settling, orientation of the leaf was not a factor.

To avoid the formation of air bubbles within the cake, which would impede the flow of filtrate, all batches of slurry were prepared with deaerated sugar solutions. We found, as did others (7), that inconsistent data were obtained if this was not done, owing to gas coming out of solution under vacuum. Deaeration was accomplished by boiling the filtrate under high vacuum, taking care that air did not reenter the liquid.

In operation the modified test leaf and filtrate delivery line were first filled with clear filtrate, and the inverted leaf immersed in the slurry. A worm gear and rack from an elevator type of tripod made it possible to smoothly raise and lower the test leaf. The filter cloth was immersed to a depth of about 2 in. and the leaf lowered during filtration as the slurry level dropped. Before the start of each test the system was evacuated to the desired vacuum. Opening the clamp on the delivery line signified the start of the run. Because of the well-suspended nature of the slurry, use of the stirrer was unnecessary except for the longer runs, and then only for short intervals at low speeds. Turbulence could not therefore interfere with cake growth.

As a general rule, it was found that about five different cakes of varying size could be created from each 35-liter batch of working slurry. These cakes were formed at constant vacua and ranged in size from a maximum of about $\xi_o = 2.0$ down to a minimum of about $\xi_o = 0.2$, the largest cake being formed first. Most of these runs were made using a 200 × 200 stainless steel wire-mesh filter cloth with a 4-in. exposed diameter. Several additional tests were carried out using an exposed cloth diameter of 4/3-in. This latter technique enabled us to obtain dimensionless thicknesses ξ_o up to 6.5 without actually using a larger filter leaf. The wire-mesh cloth was chosen because of its very small resistance to flow. Several runs were also made with a high-resistance cotton-twill cloth to study the effect of cloth resistance. In order to study compressibility effects, constant pressure drops of 5, 15, and 25 in. Hg were imposed across the cakes by controlling the vacuum in the receiving vessel. During filtration simultaneous readings of filtrate level and time were taken every 0.5 liters.

After a cake of desired size was grown, the filter leaf was quickly raised out of the slurry and turned upright. Air was allowed to pass through the cake for a short time to remove some of the entrapped filtrate. Each cake seemed to be symmetrical and well-formed; no cracks or other imperfections were observed. In all cases the filtrate was exceptionally clear.

In order to obtain profiles of the cakes, photographs were taken of each hemispheroid immediately after filtration. Care was taken to avoid any distortions or parallex problems. High-contrast photocopy film was used. Profile views were obtained from three different horizontal directions. Slides made from these negatives were projected onto a rigid screen from which a duplicate image of the spheroid could be drawn onto a piece of tracing paper. The three views of each cake were then superimposed on a viewing glass and a master profile was drawn on graph paper by visually averaging the three outlines. In almost every case the three separate views coincided exactly, implying perfect circular symmetry.

After the photographs were taken the cakes were carefully removed from the leaf and weighed. In order to obtain dry weights of the Lucite powder, the dissolved sugar was removed by washing the solids with hot water over a large Buchner funnel fitted with double layers of extra fine filter paper. When the wash water showed a negative reaction to the Fehling's solution test, the Lucite particles were placed in an oven, dried, and weighed.

The wet volumes of the spheroidal cakes were obtained by graphical integration of their respective profiles in accordance with Simpson's rule for bodies of revolution. These cake volumes, in conjunction with the corresponding dry cake weights and true solids density of the Lucite powder, enabled the cake porosities to be determined from the formula

$$\epsilon = 1 - \frac{\text{Dry Cake Weight}}{\rho_s \text{ (Wet Cake Volume)}} \tag{11}$$

The exact slurry concentration was obtained from a material balance using the weight of dry cake and filtrate volume, corrected for fluid originally in the delivery line and filtrate remaining in the cake.

ONE-DIMENSIONAL TESTS

Conventional one-dimensional tests were conducted to obtain quantitative measurements of both cake compressibility and cloth resistance. This necessitated the collection of cumulative filtrate volume vs. form time data.

The experimental setup for one-dimensional filtration was essentially the same as that used in the three-dimensional tests, the only major differences being in the filter leaf and receiving vessel. The filter cell consisted of a cell body, a screen support on which the filter cloth rested, and a cylindrical sleeve which laterally constrained the cake. It was modified from an aluminum filter funnel obtained from the Gelman Instrument Company (Figure 4). Precautions were taken to prevent the filter cell from leaking around the collar. The receiving vessel was a 1,000-ml. graduated cylinder whose top had been cut off just below the pouring spout. This provided a smooth circular opening in which to place a two-hole rubber stopper. The delivery line between filter cell and filtrate receiver had a large diameter (½ in.) and short length, thereby making line pressure drop negligible.

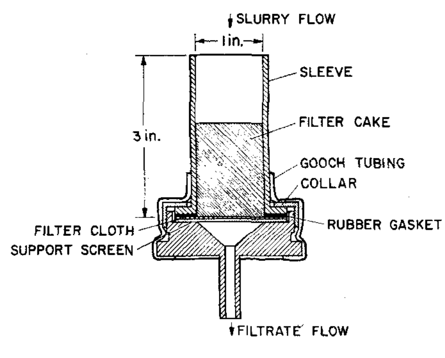

Fig. 4. Filter cell for one-dimensional filtration

In order to obtain consistent and reproducible results for filtrate volume vs. time data, the time was measured to a fraction of a second. This was accomplished as follows: As the accumulated filtrate increased by increments of 50 ml., a buzzer was manually actuated and the sound fed into a tape recorder. Upon playback it was possible to measure the time between oscillations with the necessary precision.

Tests were carried out using approximately a 2% by weight Lucite-sugar solution slurry. Although, theoretically, the cake characteristics, specific cake resistance, and porosity are independent of filtrate viscosity, duplicate tests were also performed using distilled water in place of the 40% sugar solution. However, a settling problem was encountered, which was resolved by resorting to higher mixing speeds. In general, more consistent results were obtained with the sugar solution, not only because the slurry was of more uniform concentration, but also because the time required for each corresponding experiment was increased due to the slower flow rate of the high viscosity filtrate. This extended action resulted in a large increase in experimental accuracy of the time-volume measurements.

In order to obtain cake compressibility data, runs were made at various constant pressures by applying vacua of 5, 10, 15, 20, and 25 in. Hg vacuum. Several types of filter cloths were used: a 200 × 200 stainless steel mesh cloth, a polyethylene cloth, and a cotton-twill cloth. The values ultimately obtained for the filter resistance of the clean wire mesh were much less reproducible than for the other cloths. The wire-mesh cloth was of the same type used in the three-dimensional tests and, as previously mentioned, was selected for the majority of tests because of the low values of its resistance.

After the cake had reached a thickness of about 2-in., the run was concluded by raising the test cell out of the slurry, thus draining the delivery line and dewatering the cake. The cake height was measured and the sleeve containing the cake removed. The cake was expelled into a weighing bottle together with any scrapings from the filter cloth. Dry cake weight was obtained after washing the cake with hot water and oven drying, as in the three-dimensional experiments.

RESULTS AND DISCUSSION

One-Dimensional Filtration

The more significant data obtained from the one-dimensional, Lucite-sugar water runs are recorded in Table 1. The calculated values of specific cake resistance and filter cloth resistance were obtained as follows: Plots of the instantaneous $\Delta t/\Delta V$ vs. $\overline{V}$ values were prepared for the stainless steel cloth (Figure 5). [Corresponding plots were obtained (*4*) for the other cloths, but are not shown here.] Since the plots were linear, it was a straightforward matter to compute the specific cake resistance α and cloth resistance R_m from the differential form of the one-dimensional filtration equation (*1*)

$$\frac{dt}{dV} = \frac{\mu}{A\,\Delta p\, g_c}\left[\frac{\rho\, w\, \alpha}{A\,(1-mw)}V + R_m\right] \qquad (12)$$

by determining the slope and intercept.

The constancy of the ϵ and α values tabulated in Table 1 indicated that the cakes were indeed incompressible over the fivefold range of pressure drops investigated. Note, too, that the stainless steel wire mesh cloth had the lowest resistance of all the filter media investigated, being several hundred times smaller than for the cotton cloth. The negative values reported in a few cases for the resistance of the stainless steel cloth are not significant.

Table 1. Experimental Results from One-Dimensional Filtration Tests.* Lucite-Sugar Solution Slurry

Pressure drop Δp, in. Hg	Slurry concentration w	Moisture ratio m	Porosity ϵ	Specific cake resistance α, ft./lb.	Filter cloth resistance R_m, ft.$^{-1}$
		Polyethylene cloth			
25	0.0385	1.259	0.403	4.58×10^8	3.09×10^8
20	0.0370	1.254	0.393	4.59	1.93
15	0.0371	1.253	0.398	4.53	5.50
10	0.0367	1.268	0.400	4.43	3.84
5	0.0364	1.236	0.398	4.63	3.42
	Averages:	1.254	0.398	4.55×10^8	3.56×10^6
		Stainless steel wire 200 mesh cloth			
25	0.0179	1.247	0.396	4.80×10^8	-7.06×10^7
20	0.0180	1.250	0.395	4.81	-2.17
15	0.0179	1.250	0.400	4.59	1.63
10	0.0178	1.277	0.398	4.81	-4.29
5	0.0178	1.273	0.397	4.88	-0.30
	Averages:	1.259	0.397	4.78×10^8	-2.44×10^7
		Cotton-twill cloth			
20	0.0210	1.262	0.402	4.75×10^8	7.85×10^9
15	0.0208	1.252	0.396	4.82	6.98
	Averages:	1.257	0.399	4.79×10^8	7.41×10^9
Grand averages:		1.257	0.398	4.67×10^8	

* Filter cloth area = 5.51×10^{-3} sq. ft.

Rather, as is apparent from the nearness of the intercepts to the origin in Figure 5, these values merely reflect the minuteness of the cloth resistance compared with that of the cake, and hence the concomitant difficulty in measuring it with any degree of precision.

Not presented here are data obtained for comparable one-dimensional experiments performed with distilled water in place of the 40% sugar solution (*4*). One purpose of these auxiliary tests was to compare the results with another study in which cake resistance and porosity values were obtained for a Lucite-water slurry, using both conventional filtration apparatus and a compressibility cell (*7*). The latter investigation showed that the Lucite powder cake is virtually incompressible over the same range of Δp values investigated here. In that study the average values of specific cake resistance and cake porosity were reported as 6.63×10^8 ft./lb. and 0.394, respectively. Our distilled water results were in excellent agreement with these, being 6.33×10^8 ($\pm$ 5%) ft./lb. and 0.396 ($\pm$ 2%), respectively. This favorable comparison gave a certain degree of confidence in both the techniques used and results obtained in the remainder of the investi-

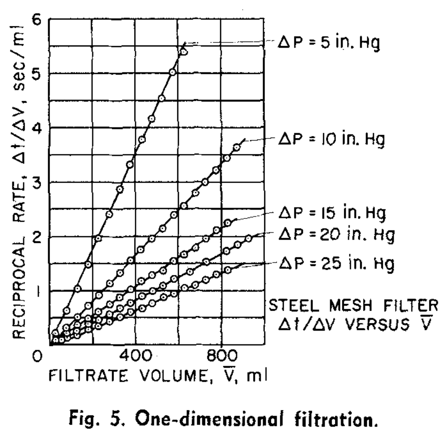

Fig. 5. One-dimensional filtration. Lucite-sugar solution slurry.

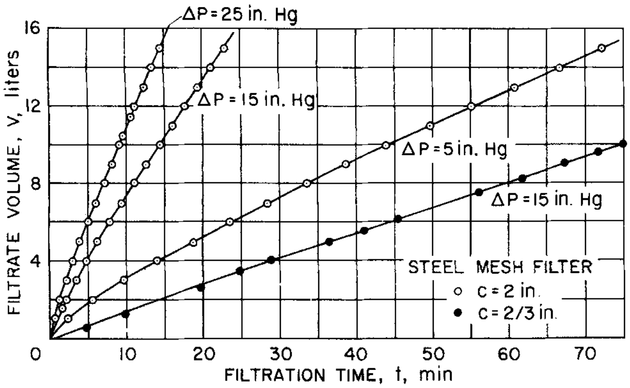

Fig. 6. Filtrate volume vs. filtration time. Three-dimensional filtration.

gation. It should be carefully noted that these specific resistances are significantly higher than the mean value of 4.67×10^8 obtained with the sugar solutions, showing that even for incompressible cakes the properties of the supernatant influence the ultimate cake properties. Since the sugar solutions were filtered at a much smaller rate owing to their higher viscosity, it may be that, in general, higher filtration rates result in larger cake resistances—all other things being equal.

Three-Dimensional Filtration

A total of thirty-four runs was made with the Lucite-sugar water slurries in the series of three-dimensional experiments. Each of these produced a different spheroidal cake under varying conditions of pressure drop, cloth area, cloth material, and filtration time. All but five runs were made with the steel mesh cloth; cotton was used in the remainder. The overall data obtained in each of these experiments are tabulated in Table 2.* Values of the moisture ratio m varied widely because the cakes were partially purged with air at the termination of a run, thus removing varying fractions of filtrate. Cake porosities remained essentially constant over the entire Δp range, and were within 2 percent of the corresponding values observed in the one-dimensional tests.

Cumulative filtrate volume vs. time experiments were made for the five runs indicated by an asterisk in Table 2, corresponding to growth of five of the largest spheroids formed. a values varied from 10.7 to 12.6 cm. and b values from 10.2 to 10.6 cm. Four of these runs were made with the stainless steel mesh and one with the cotton-twill cloth (Δp = 15 in. Hg, c = 2 in. for the latter). These V vs. t data are plotted arithmetically in Figure 6, save for the run with the cotton cloth. As can be observed, the curves exhibit an almost linear behavior at long times, in

* Table 2 and Figures 11 and 12 have been deposited as document 8402 with the American Documentation Institute, Photoduplication Service, Library of Congress, Washington 25, D. C. and may be obtained for $1.25 for photoprints or $1.25 for 35-mm. microfilm.

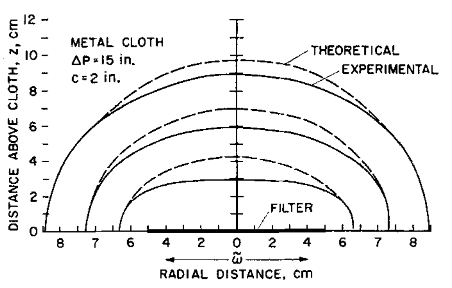

Fig. 7. Experimental vs. theoretical cake profiles. Low ξ_0 range.

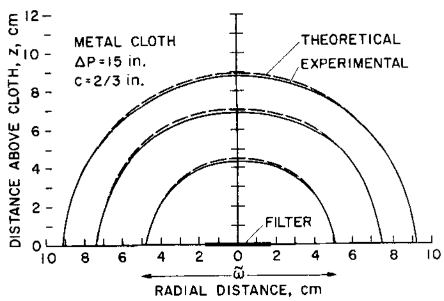

Fig. 8. Experimental vs. theoretical cake profiles. High ξ_0 range.

conformity with the asymptotic prediction of Equation (7). The single trial with the high resistance cotton cloth (not shown) did not achieve any significant degree of linearity by the end of the run owing to the large cloth resistance. According to Equation (7) the slope of the linear portions of these curves is $4ck\Delta pg_c/\mu$. This relation was used to compute the cake permeability k for each of the four runs with the stainless steel mesh. At pressure drops of 5, 15, and 25 in. Hg, and with c = 2 in., values obtained for the cake permeability were 5.79, 5.86, and 5.80×10^{-11} sq.ft., respectively. For Δp = 15 in. Hg and c = 2/3 in. the permeability obtained was 4.21×10^{-11} sq.ft. Together, these four experiments lead to an average value of $k = 5.42 \times 10^{-11}$ sq.ft. The permeability and specific cake resistance are related by the expression (3)

$$k = \frac{1}{\alpha(1-\epsilon)\rho_s} \tag{13}$$

The average α value obtained in the one-dimensional experiments thus leads to a permeabiltiy of 4.82×10^{-11} sq.ft. This is roughly 10% lower than the average permeability of 5.42×10^{-11} sq.ft. obtained in the three-dimensional experiments. Considering the very different nature of the experiments and the idealizations implicit in the analyses, one can regard this agreement as very satisfactory.

According to the general theory (3), the outer surface of the cake is described by Equations (9) and (10). If the filtration were to be interrupted at various times, and the spheroidal cake profiles recorded, a family of confocal semiellipses should result. Figures 11 and 12* depict the ellipsoidal-like profiles for two series of runs made under identical process conditions. In the first of these figures, a 2-in. radius filter cloth was used. ξ_o values varied from 0.2 to 2.1. In the second the radius was reduced to 2/3 in.

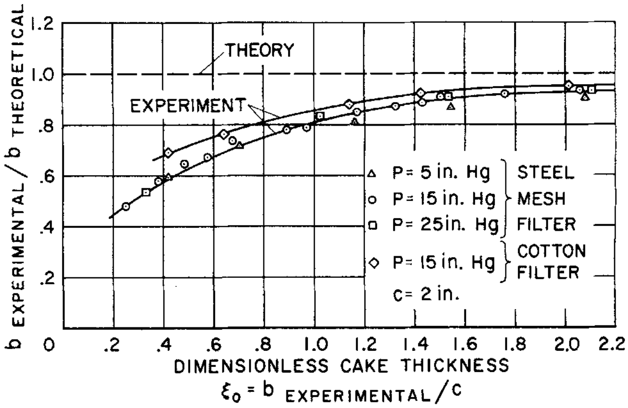

Fig. 9. Ratio of experimental to theoretical cake thickness. Low ξ_0 range.

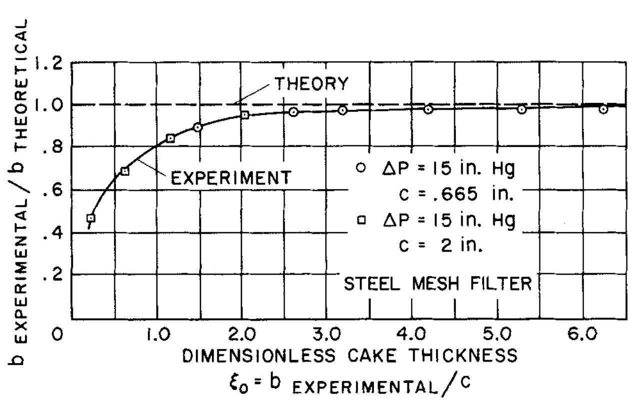

Fig. 10. Ratio of experimental to theoretical cake thickness. High ξ_0 range.

Here, ξ_o values varied from 1.5 to 6.2. Each profile represented a cake grown for a different period of time and was characterized by its dimensionless cake thickness ξ_o. Figures 7 and 8 compare several experimental profiles with their theoretical counterparts. The latter were obtained by utilizing the experimental value for a (the major semiaxis) and calculating the elliptical surface from Equations (9) and (10). Notice that in each case the observed cake tends to be significantly flatter than the ideal cake, although the divergence decreases with increasing cake thickness. Note, too, that in Figure 8, where large dimensionless cake thicknesses were achieved, the profiles are almost semicircular as the theory predicts.

A quantitative measure of departure from ideal shape is furnished by taking the ratio of experimental to theoretical cake thickness and by plotting it as a function of the dimensionless cake thickness. This was done in Figures 9 and 10, where it can be seen that the experimental results asymptotically converge to the theoretical value of unity; for values of ξ_o greater than 3.0 the results are within 5% of this value. However, at the smaller values of ξ_o there is serious disagreement between theoretical and experimental predictions. The actual cake is significantly flatter than the ideal one.* Compressibility of the cake cannot be invoked to explain this phenomenon, for the cakes have been shown to be incompressible in the pressure range of interest. Moreover, as is clear from Figure 9, departure from the theory is independent of the imposed pressure drop. Neither can nonzero cloth resistance be invoked to explain the discrepancy, for as is also clear from Figure 9 the results for the cotton-twill cloth approach the theory more closely than do those for the wire mesh cloth despite the fact the former offers more than one hundred times the resistance of the latter in one-dimensional experiments.

The explanation of the discrepancy may be more fundamental. At the foundation of the present theory lies Darcy's law. Inasmuch as this law was originally formulated for the slow, steady state flow of fluids through fixed porous beds, it can be legitimately questioned whether it applies equally well to an unsteady state filtration process, where there is continuous deposition of particles. Furthermore, the high flow rates encountered at the start of the experiment may be above the range for which this linear flow law is known to be valid. While it is true that Darcy's law has been shown to be essentially valid in one-dimensional filtration theory, the cake shape has never before played a significant role in proving the consistency of the theory.

* Subsequent to the completion of this work in 1962 we were informed that similar observations of cake flatness were made by Shirato et al. Unfortunately, the work was originally available to us only in the form of an abstract (5), so no attempts are made here to compare our results quantitatively.

In any future work it is recommended that the cake shapes be determined in situ, before removal of the test leaf from the slurry. This would eliminate the possibility of the cake deforming under its own weight and of irreversible cake deformation arising from the pressure difference across the cake during the brief dewatering stage. Though such effects are believed to be negligible, it would be desirable to eliminate these possibilities before concluding that Darcy's law was inapplicable.

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NOTATION

a = equatorial radius of spheroidal cake, ft.
A = area of circular filter cloth, sq.ft.
b = polar radius (thickness) of spheroidal cake, ft.
c = radius of circular filter cloth, ft.
g_c = 32.16 (lb.$_m$)(ft.)/(lb.$_f$)(sec.^{2})
k = filter cake permeability, sq.ft.
m = moisture ratio; weight ratio of wet cake to dry, washed cake
p = instantaneous local pressure in cake, lb.$_f$/sq.ft.
Δp = $p_o - p_i$ = pressure drop across filter cake plus cloth, lb.$_f$/sq.ft.
q = instantaneous volumetric flow rate of filtrate through cake, cu.ft./sec.
R_m = cloth resistance, ft.$^{-1}$
t = time from start of filtration, sec.
T = dimensionless time from start of filtration
$\vec{v}$ = instantaneous local superficial velocity vector of filtrate in cake, ft./sec.
V = cumulative filtrate volume, cu.ft.
$\mathcal{V}$ = dimensionless volume of filtrate collected up to dimensionless time T
w = weight fraction of insoluble solids in original slurry
x, y = Cartesian coordinates measured in the plane of the filter cloth, ft.
z = Cartesian coordinate measured normal to filter cloth, ft.

Greek Letters

α = specific cake resistance, ft./lb.$_m$
ϵ = porosity of filter cake
μ = filtrate viscosity, lb.$_m$/(ft.)(sec.)
ξ_o = dimensionless spheroidal cake thickness = b/c
ρ = density of filtrate, lb.$_m$/cu.ft.
ρ_s = true density of dry, washed solids, lb.$_m$/cu.ft.
$\tilde{\omega}$ = $(x^2 + y^2)^{1/2}$ = cylindrical distance, ft.

Subscripts

expt. = experimental
i = value at inner surface of filter cake or cloth
o = value at outer surface of filter cake in contact with slurry
theor. = theoretical

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On Finite Amplitude Roll Cell Disturbances in a Fluid Layer Subjected to Heat and Mass Transfer

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The system consists of a thin, transversely infinite horizontal fluid layer initially in mechanical but not thermostatic or speciestatic equilibrium. The top and bottom surfaces of the layer are assumed to be unconstrained, and the stability of the system to buoyancy driven finite amplitude roll cell disturbances is investigated. The system is capable of exhibiting both a subcritical stationary and oscillatory instability. The average vertical transport of heat and mass is computed and it is found that the convective transport associated with a stationary type of instability is much greater than that corresponding to an oscillatory instability.

In engineering, meteorology, biology, and other branches of science, many of the complex problems which are encountered involve the interaction of transport and transformation processes which are coupled at the macroscopic level. In many such dynamic systems the stability of the system is the object of prime interest. Analytical treatments of such systems have proceeded at a slow rate owing to the inherent complexity of a reasonable mathematical description. Moreover, most of the available stability analyses deal with the stability of the system to infinitesimal disturbances and utilize the simplification of a linearized mathematical model. The stability of the system to infinitesimal disturbances may, or may not, provide an adequate description of the stability of the system to disturbances of finite amplitude (see reference 9). It does not in the system to be considered here.

The present analysis deals with the convective instability of a system undergoing heat and mass transfer to one form of finite amplitude disturbance. The distortion of the mean temperature and concentration fields by the disturbance, the configuration of the local temperature and concentration fields, and the transport of heat and mass at a finite amplitude roll cell equilibrium state are considered; the prediction of the cellular pattern with the greatest tendency to form in the physical system and the tendency for a change in cell size are not considered. Not only does the analysis help to fill a gap in one area of convective instability in general, but at the same time it illustrates such features as a subcritical instability, a bifurcation point where the regime of the developed instability undergoes a transition from a stationary instability to an oscillatory instability, and the poor convective transport characteristics of an oscillatory instability.

DESCRIPTION OF THE SYSTEM

The system to be investigated is a transversely infinite horizontal fluid layer initially in mechanical but not thermostatic or speciestatic equilibrium, with heat and mass transfer across the layer and concomitantly a potentially unstable density gradient. The only force presumed to exist in the initial quiescent system is the force of gravity g, which is constant to very good approximation; consequently, the condition for mechanical equilibrium is

$$(\nabla \rho) \times g = 0 \tag{2.1}$$

By thermostatics $\rho = f(\theta^*, \Lambda^*, p^*)$ and a truncated Taylor expansion leads to

$$\rho = \rho_r \{1 + \gamma(\theta^* - \theta_r^*) + \beta(\Lambda^* - \Lambda_r^*)\} \tag{2.2}$$

Here the first-order pressure variation of the density is assumed to be negligible. Substitution of Equation (2.2) into (2.1) leads to the equilibrium criterion, which is appropriate for the present system.